NUCLEAR SCIENCE AND ENGINEERING: **174**, 150–162 (2013)

# Multilevel NDA Methods for Solving Multigroup Eigenvalue Neutron Transport Problems

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*Received April 30, 2012*
*Accepted August 10, 2012*

**Abstract** – *The nonlinear diffusion acceleration (NDA) method is an efficient and flexible transport iterative scheme for solving reactor-physics problems. This paper presents a fast iterative algorithm for solving multigroup neutron transport eigenvalue problems in one-dimensional slab geometry. The proposed method is defined by a multilevel system of equations that includes multigroup and effective one-group low-order NDA equations. The eigenvalue is evaluated in an exact projected solution space of the smallest dimensionality. Numerical results that illustrate the performance of the new algorithm are demonstrated.*

## I. INTRODUCTION

One of the common iterative approaches for solving the transport equation is the nonlinear diffusion acceleration (NDA) method.[1,2] It is an efficient algorithm for acceleration of within-group transport iterations for reactor-physics calculations. The NDA method is a nonlinear projection-iteration method.[3] The low-order NDA (LONDA) equations are defined for the moments of the transport solution and consist of (a) the particle balance equation and (b) the first-moment equation in a form of a generalized Fick's law with a special consistency term. The high-order and low-order equations of the method are coupled by means of an exact closure.

Let us consider the one-speed $k$-eigenvalue transport problem for one-dimensional slab geometry with isotropic scattering:

$$\mathcal{L}\psi(x,\mu) = \frac{1}{2}\left(\Sigma_s(x) + \frac{1}{k}\,\nu_f(x)\Sigma_f(x)\right) \times \int_{-1}^{1} \psi(x,\mu')\,d\mu' \ , \quad -1 \le \mu \le 1,\ 0 \le x \le X \tag{1}$$

and

$$\mathcal{L}\psi \equiv \mu\,\frac{\partial\psi}{\partial x} + \Sigma_t\psi \ , \tag{2}$$

with the following boundary conditions:

$$\psi(0,\mu) = \psi(0,-\mu) \quad \text{for } \mu > 0 \ ,$$
$$\psi(X,\mu) = 0 \quad \text{for } \mu < 0 \ . \tag{3}$$

The NDA method can be formulated in a general differential form by means of the nonlinear system of equations given by

$$\mathcal{L}\psi = \frac{1}{2}\left(\Sigma_s + \frac{1}{k}\,\nu_f\Sigma_f\right)\phi \ , \tag{4}$$

$$\frac{dJ}{dx} + \left(\Sigma_t - \Sigma_s - \frac{1}{k}\,\nu_f\Sigma_f\right)\phi = 0 \ , \tag{5}$$

$$J = -\frac{1}{3\Sigma_t}\,\frac{d\phi}{dx} + \tilde{D}\phi \ , \tag{6}$$

and

$$J(0) = 0 \ , \quad J(X) = F_X\,\phi(X) \ , \tag{7}$$

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where

$$\tilde{D} = \frac{1}{\int_{-1}^{1} \psi \, d\mu} \left( \int_{-1}^{1} \mu\psi \, d\mu + \frac{1}{3\Sigma_t} \frac{d}{dx} \left[ \int_{-1}^{1} \psi \, d\mu \right] \right) \tag{8}$$

is the factor that defines a nonlinear closure of the high-order transport equation (4) and the LONDA equations (5) and (6) and where

$$F_X = \frac{\int_{-1}^{1} \mu\psi(X,\mu) \, d\mu}{\int_{-1}^{1} \psi(X,\mu) \, d\mu} \tag{9}$$

is the boundary factor that is used to formulate the boundary conditions for the LONDA equations. Here, $\phi = \int_{-1}^{1} \psi \, d\mu$ is the scalar flux and $J = \int_{-1}^{1} \mu\psi \, d\mu$ is the current. A linearization of the LONDA equations in the vicinity of the solution leads to equations that have a form similar to one of the low-order equations of the first-moment method.[3,4] Thus, the first-moment method can be considered as a linear version of the NDA method.

The system of NDA equations (4) through (7) can be solved by means of the following iteration process[1,2]:

$$\mathcal{L}\psi^{(s+1/2)} = \frac{1}{2} \left( \Sigma_s + \frac{1}{k^{(s)}} \nu_f \Sigma_f \right) \phi^{(s)} \ , \tag{10}$$

$$\phi^{(s+1/2)} = \int_{-1}^{1} \psi^{(s+1/2)} \, d\mu \ , \tag{11}$$

$$J^{(s+1/2)} = \int_{-1}^{1} \mu\psi^{(s+1/2)} \, d\mu \ , \tag{12}$$

$$\tilde{D}^{(s+1/2)} = \frac{1}{\phi^{(s+1/2)}} \left( J^{(s+1/2)} + \frac{1}{3\Sigma_t} \frac{d\phi^{(s+1/2)}}{dx} \right) \ , \tag{13}$$

$$F_X^{(s+1/2)} = \frac{J^{(s+1/2)}(X)}{\phi^{(s+1/2)}(X)} \ , \tag{14}$$

$$\frac{dJ^{(s+1)}}{dx} + \left( \Sigma_t - \Sigma_s - \frac{1}{k^{(s+1)}} \nu_f \Sigma_f \right) \phi^{(s+1)} = 0 \ , \tag{15}$$

$$J^{(s+1)} = -\frac{1}{3\Sigma_t} \frac{d\phi^{(s+1)}}{dx} + \tilde{D}^{(s+1/2)} \phi^{(s+1)} \ , \tag{16}$$

and

$$J^{(s+1)}(0) = 0 \ , \quad J^{(s+1)}(X) = F_X^{(s+1/2)} \phi^{(s+1)}(X) \ . \tag{17}$$

This algorithm can be interpreted as a fixed-point iteration method.[5] Another way to solve the NDA equations is to apply the Jacobian-free Newton-Krylov method.[6]

Usually, the NDA method is formulated directly in a discrete form. The NDA method can be applied to any transport discretization scheme. The spatial grids for the transport equation and LONDA equations (5) and (6) can be different. For example, the LONDA problem can be applied to pin-cell regions, whereas the transport equation itself is solved on a mesh that resolves the structure inside pin cells.[2,6] Then, the discrete balance equation (5) in each coarse spatial cell relates the particle net leakage rate across the pin-cell boundary with absorption and fission production rates in the given pin cell. The discrete first-moment equation (6) is formulated at pin-cell interfaces. It provides a relationship between the net leakage rate at these interfaces and the average scalar fluxes in neighboring pin cells. In this paper, the LONDA equations are considered on grids that are used for the transport equation.

The multigroup transport $k$-eigenvalue problem is given by

$$\begin{aligned} \mu \frac{\partial}{\partial x} \psi_g(x,\mu) + \Sigma_{t,g}(x)\psi_g(x,\mu) &= \frac{1}{2} \sum_{g'=1}^{G} \Sigma_{s,g'\to g}(x) \int_{-1}^{1} \psi_{g'}(x,\mu) \, d\mu \\ &\quad + \frac{1}{2k} \chi_g(x) \sum_{g'=1}^{G} \nu_{f,g'}(x) \Sigma_{f,g'}(x) \\ &\quad \times \int_{-1}^{1} \psi_{g'}(x,\mu) \, d\mu \ , \\ -1 \le \mu \le 1, \ 0 \le x \le X, \ g &= 1, \ldots, G \end{aligned} \tag{18}$$

and

$$\begin{aligned} \psi_g(0,\mu) &= \psi_g(0,-\mu) \quad \text{for } \mu > 0 \ , \\ \psi_g(X,\mu) &= 0 \quad \text{for } \mu < 0 \ , \end{aligned} \tag{19}$$

where standard notations are used. The NDA method can be used directly to solve multigroup transport problems by applying it to the transport equation in each group. It is defined as follows:

$$\begin{aligned} \mu \frac{\partial \psi_g}{\partial x} + \Sigma_{t,g}\psi_g &= \frac{1}{2} \sum_{g'=1}^{G} \Sigma_{s,g'\to g} \phi_{g'} \\ &\quad + \frac{\chi_g}{2k} \sum_{g'=1}^{G} \nu_{f,g'} \Sigma_{f,g'} \phi_{g'} \ , \end{aligned} \tag{20}$$

$$\begin{aligned} \frac{dJ_g}{dx} + \Sigma_{t,g}\phi_g &= \sum_{g'=1}^{G} \Sigma_{s,g'\to g} \phi_{g'} \\ &\quad + \frac{\chi_g}{k} \sum_{g'=1}^{G} \nu_{f,g'} \Sigma_{f,g'} \phi_{g'} \ , \end{aligned} \tag{21}$$



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$$J_g = -\frac{1}{3\Sigma_{t,g}} \frac{d\phi_g}{dx} + \tilde{D}_g \phi_g \ , \tag{22}$$

$$J_g(0) = 0 \ , \quad J_g(X) = F_{X,g}\phi_g(X) \ , \tag{23}$$

$$\tilde{D}_g = \frac{1}{\int_{-1}^{1} \psi_g \, d\mu} \times \left( \int_{-1}^{1} \mu\psi_g \, d\mu + \frac{1}{3\Sigma_{t,g}} \frac{d}{dx}\left[ \int_{-1}^{1} \psi_g \, d\mu \right] \right) \ , \tag{24}$$

and

$$F_{X,g} = \frac{\int_{-1}^{1} \mu\psi_g(X,\mu) \, d\mu}{\int_{-1}^{1} \psi_g(X,\mu) \, d\mu} \ . \tag{25}$$

In this case, the multigroup LONDA equations are similar in form to the multigroup $P_1$ equations. The method of power iterations[3] is used to solve the multigroup LONDA equations (21), (22), and (23). The estimation of the eigenvalue is obtained as a result of solving the multigroup LONDA problem. In this paper, to accelerate the convergence of multigroup transport iterations, it is proposed to formulate a one-group (grey) low-order transport problem based on the LONDA equations.[7,8]

The remainder of this paper is organized as follows. In Sec. II we present the two-level NDA (TLNDA) method in a discrete form for solving multigroup transport problems. In Sec. III we formulate the multilevel NDA (MLNDA) method. We present the numerical results in Sec. IV. We conclude with a summary in Sec V.

## II. THE TLNDA METHOD

The NDA method for solving the multigroup transport problems is defined by a high-order multigroup problem for the transport discretization scheme and a multigroup problem for the LONDA equations. To discretize the transport equation, we use the step characteristic method. The multigroup high-order transport problem is given by

$$\mu_m(\psi_{m,g,j+1/2} - \psi_{m,g,j-1/2}) + \Sigma_{t,g,j}\Delta x_j \psi_{m,g,j} = \frac{\Delta x_j}{2}\left( \sum_{g'=1}^{G} \Sigma_{s,g'\to g,j}\phi_{g',j} + \frac{\chi_{g,j}}{k} \sum_{g'=1}^{G} \nu_{f,g',j}\Sigma_{f,g',j}\phi_{g',j} \right) \ , \tag{26}$$

$$\psi_{m,g,j} = \alpha_{m,g,j}\psi_{m,g,j-1/2} + (1-\alpha_{m,g,j})\psi_{m,g,j+1/2} \ , \tag{27}$$

$$\alpha_{m,g,j} = \frac{1}{\tau_{m,g,j}} - \frac{1}{e^{\tau_{m,g,j}} - 1} \ , \quad \tau_{m,g,j} = \frac{\Sigma_{t,g,j}\Delta x_j}{\mu_m} \ , \tag{28}$$

$$j = 1,\ldots,N, \ m = 1,\ldots,M, \ g = 1,\ldots,G \ ,$$

$$\psi_{m,g,1/2} = \psi_{m',g,1/2} \ , \quad m : \mu_m > 0 \quad \text{where } \mu_{m'} = -\mu_m \ , \tag{29}$$

and

$$\psi_{m,g,N+1/2} = 0 \ , \quad m : \mu_m < 0 \ , \tag{30}$$

where $\psi_{g,m,j-1/2}$ and $\psi_{g,m,j}$ are the cell-edge and cell-average group angular fluxes, respectively,

$$\Delta x_j = x_{j+1/2} - x_{j-1/2} \ , \quad j = 1,\ldots,N \ , \quad \Delta x_0 = \Delta x_{N+1} = 0 \ . \tag{31}$$

Now, the NDA method is applied to the discretized multigroup transport equations (26), (27), and (28). Let us introduce the group scalar flux and current calculated from the group angular flux by means of quadrature sums:

$$\tilde{\phi}_{g,j} = \sum_{m=1}^{M} \psi_{m,g,j} w_m \ , \quad j = 1,\ldots,N \ , \tag{32}$$

$$\tilde{\phi}_{g,0} = \sum_{m=1}^{M} \psi_{m,g,1/2} w_m \ , \quad \tilde{\phi}_{g,N+1} = \sum_{m=1}^{M} \psi_{m,g,N+1/2} w_m \ , \tag{33}$$

and

$$\tilde{J}_{g,j-1/2} = \sum_{m=1}^{M} \mu_m \psi_{m,g,j-1/2} w_m \ , \quad j = 1,\ldots,N+1 \ . \tag{34}$$



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These quantities are utilized to define the NDA consistency term,

$$\tilde{D}_{g,j+1/2} = \frac{\tilde{J}_{g,j+1/2} + D_{g,j+1/2}\dfrac{(\tilde{\phi}_{g,j+1} - \tilde{\phi}_{g,j})}{\Delta x_{j+1/2}}}{0.5(\tilde{\phi}_{g,j+1} + \tilde{\phi}_{g,j})} \ , \tag{35}$$

where

$$\Delta x_{j+1/2} = \frac{1}{2}\left(\Delta x_j + \Delta x_{j+1}\right) \ , \tag{36}$$

$$D_{g,j+1/2} = \frac{1}{3\Sigma_{t,g,j+1/2}} \ , \tag{37}$$

and

$$\Sigma_{t,g,j+1/2} = \frac{\Sigma_{t,g,j}\Delta x_j + \Sigma_{t,g,j+1}\Delta x_{j+1}}{\Delta x_j + \Delta x_{j+1}} \ , \tag{38}$$

and they define the boundary factor,

$$F_{g,X} = \frac{\tilde{J}_{g,N+1/2}}{\tilde{\phi}_{g,N+1}} \ , \tag{39}$$

that is used to formulate the low-order boundary conditions. Note that in the case of anisotropic scattering, $D_g$ is defined by the group transport cross section. The resulting multigroup LONDA equations have the following form:

$$\begin{aligned} J_{g,j+1/2} - J_{g,j-1/2} + \Sigma_{t,g,j}\Delta x_j \phi_{g,j} &= \Delta x_j \sum_{g'=1}^{G} \Sigma_{s,g'\to g,j}\phi_{g',j} \\ &\quad + \Delta x_j \frac{\chi_{g,j}}{k}\sum_{g'=1}^{G}\nu_{f,g',j}\Sigma_{f,g',j}\phi_{g',j} \ , \end{aligned} \tag{40}$$

$$j = 1,\ldots,N \ ,$$

$$\begin{aligned} J_{g,j+1/2} &= -D_{g,j+1/2}\frac{\phi_{g,j+1} - \phi_{g,j}}{\Delta x_{j+1/2}} \\ &\quad + \frac{1}{2}\tilde{D}_{g,j+1/2}(\phi_{g,j+1} + \phi_{g,j}) \ , \end{aligned} \tag{41}$$

$$j = 0,\ldots,N \ ,$$

and

$$J_{g,1/2} = 0 \ , \quad J_{g,N+1/2} = F_{g,X}\phi_{g,N+1} \ , \tag{42}$$

where $\phi_{g,0}$ and $\phi_{g,N+1}$ are the group scalar fluxes at the boundaries and $\phi_{g,j}$ $(j = 1,\ldots,N)$ are the cell-average scalar fluxes.

On each transport iteration, Eqs. (40), (41), and (42) are solved by the power iteration method with the current estimates of the consistency term (35) and boundary factor (39) to evaluate the multiplication factor and associated multigroup eigenfunction. Hereafter, we refer to the method described above as the TLNDA method. The iteration scheme of the TLNDA method is presented in Algorithm 1 in the form of a pseudo code:

*Algorithm 1:* Iteration scheme of the TLNDA method

```
s = 0
• Initialization
Set D̃_g^(1/2) = 0 and F_{g,X}^(1/2) = 0.5
Define φ_g^(0) and k^(0)
• Level 1: Transport iterations
1 if ‖φ^(s) − φ^(s−1)‖_∞ > ((1/ρ_{φ,1}) − 1)ε_1 or
    |k^(s) − k^(s−1)| > ((1/ρ_{k,1}) − 1)ε_1 then
      s = s + 1
      l = 0
      • Level 2: Multigroup low-order iterations
2     if ‖φ^(s,l) − φ^(s,l−1)‖_∞ > ((1/ρ_{φ,2}) − 1)ε_2
        or |k^(s,l) − k^(s,l−1)| > ((1/ρ_{k,2}) − 1)ε_2 then
          l = l + 1
          for g = 1 to G do
            Solve the multigroup LONDA equations (40),
              (41), and (42) to calculate φ_g^(s,l)
          end
          Calculate k^(s,l) = k^(s,l−1) ‖ν_f Σ_f φ^(s,l)‖/
            ‖ν_f Σ_f φ^(s,l−1)‖
      end
      φ_g^(s) = φ_g^(s,l) and k^(s) = k^(s,l)
      • Transport sweep
      for g = 1 to G do
        Solve the transport equation (26) to calculate
          ψ_g^(s+1/2)
      end
      for g = 1 to G do
        Calculate D̃_g^(s+1/2) and F_{g,X}^(s+1/2)
      end
  end
```

The version of the TLNDA method in which iterations at level 2, namely, the multigroup low-order iterations, are interrupted if they exceed a given number $i$ is referred to as TLNDA$(i)$. The TLNDA$(i)$ method is presented in Algorithm 2:

*Algorithm 2:* Iteration scheme of the TLNDA$(i)$ method

```
s = 0
...
• Level 1: Transport iterations
1 if ‖φ^(s) − φ^(s−1)‖_∞ > ((1/ρ_{φ,1}) − 1)ε_1
    or |k^(s) − k^(s−1)| > ((1/ρ_{k,1}) − 1)ε_1 then
      s = s + 1
      l = 0
      • Level 2: Multigroup low-order iterations
```



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```
2   | if l ≤ i and {‖φ^(s,l) − φ^(s,l−1)‖_∞ >
    |   ((1/ρ_φ,2) − 1)ε_2 or |k^(s,l) − k^(s,l−1)| >
    |   ((1/ρ_k,2) − 1)ε_2}
    | then
    |   | l = l + 1
    |   | ...
    | end
    | φ_g^(s) = φ_g^(s,l) and k^(s) = k^(s,l)
    | ...
  end
```

We note that there exists another family of two-level methods for solving transport problems, which is based on a LONDA problem applied on coarse spatial grids and involves elements of homogenization.[2]

## III. THE MLNDA METHOD

We now formulate a new algorithm that is based on the idea of the NDA method and employs effective one-group (grey) LONDA equations for the total scalar flux $\phi$ and current $J$ to accelerate multigroup low-order iterations as well as transport iterations. To couple these LONDA equations with equations of higher dimensionality, namely,

1. the multigroup LONDA equations for the group scalar flux and currents
2. the multigroup equations for the group angular flux,

and to close exactly the complete system of equations, the right sides of these multigroup equations are written in terms of the total scalar flux. The high-order multigroup transport equation for the group angular flux has the following form:

$$\mu_m(\psi_{m,g,j+1/2} - \psi_{m,g,j-1/2}) + \Sigma_{t,j}\Delta x_j \psi_{m,g,j} = \frac{\Delta x_j}{2}\left(\bar{\Sigma}_{s,g,j} + \frac{\chi_{g,j}}{k}\bar{\nu}_{f,j}\bar{\Sigma}_{f,j}\right)\phi_j \tag{43}$$

and

$$\psi_{m,g,j} = \alpha_{m,g,j}\psi_{m,g,j-1/2} + (1-\alpha_{m,g,j})\psi_{m,g,j+1/2} \ , \tag{44}$$

where the average cross sections are given by

$$\bar{\Sigma}_{s,g,j} = \frac{\sum_{g'=1}^{G} \Sigma_{s,g'\to g,j}\phi_{g',j}}{\sum_{g'=1}^{G}\phi_{g',j}} \ , \tag{45}$$

$$\bar{\Sigma}_{f,j} = \frac{\sum_{g'=1}^{G} \Sigma_{f,g',j}\phi_{g',j}}{\sum_{g'=1}^{G}\phi_{g',j}} \ , \tag{46}$$

and

$$\bar{\nu}_{f,j} = \frac{\sum_{g'=1}^{G} \nu_{f,g',j}\Sigma_{f,g',j}\phi_{g',j}}{\sum_{g'=1}^{G}\Sigma_{f,g',j}\phi_{g',j}} \ . \tag{47}$$

The multigroup LONDA equations for $\phi_g$ and $J_g$ are defined as

$$J_{g,j+1/2} - J_{g,j-1/2} + (\Sigma_{t,g,j} - \Sigma_{s,g\to g,j})\Delta x_j \phi_{g,j} = \Delta x_j \sum_{g'=1}^{g-1}\Sigma_{s,g'\to g,j}\phi_{g',j} + \Delta x_j\left(B_{up,g,j} + \frac{\chi_{g,j}}{k}\bar{\nu}_{f,j}\bar{\Sigma}_{f,j}\right)\phi_j \ , \quad j = 1,\ldots,N \ , \tag{48}$$

$$J_{g,j+1/2} = -D_{g,j+1/2}\frac{\phi_{g,j+1} - \phi_{g,j}}{\Delta x_{j+1/2}} + \frac{1}{2}\tilde{D}_{g,j+1/2}(\phi_{g,j+1} + \phi_{g,j}) \ , \quad j = 0,\ldots,N \ , \tag{49}$$

and

$$J_{g,1/2} = 0 \ , \quad J_{g,N+1/2} = F_{g,X}\phi_{g,N+1} \ , \tag{50}$$

where

$$B_{up,g,j} = \frac{\sum_{g'=g+1}^{G}\Sigma_{s,g'\to g,j}\phi_{g',j}}{\sum_{g'=1}^{G}\phi_{g',j}} \tag{51}$$

is the factor that forms the group upscattering term. To derive the effective one-group LONDA equations, we sum Eqs. (40), (41), and (42) over groups and obtain

$$J_{j+1/2} - J_{j-1/2} + \Delta x_j \sum_{g=1}^{G}(\Sigma_{t,g,j} - \Sigma_{s,g,j})\phi_{g,j} = \frac{\Delta x_j}{k}\sum_{g=1}^{G}\nu_{f,g,j}\Sigma_{f,g,j}\phi_{g,j} \ , \tag{52}$$

$$J_{g,j+1/2} = -\frac{1}{\Delta x_{j+1/2}} \times \left[\sum_{g=1}^{G} D^{+}_{g,j+1/2}\phi_{g,j+1} - \sum_{g=1}^{G} D^{-}_{g,j+1/2}\phi_{g,j}\right] \ , \tag{53}$$



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and

$$J_{1/2} = 0 \ , \quad J_{N+1/2} = \sum_{g=1}^{G} F_{g,X} \phi_{g,N+1} \ , \tag{54}$$

where

$$D^{\pm}_{g,j+1/2} = D_{g,j+1/2} \mp \frac{1}{2} \tilde{D}_{g,j+1/2} \Delta x_{j+1/2} \ . \tag{55}$$

Equation (52) leads to the effective one-group balance equation:

$$J_{j+1/2} - J_{j-1/2} + \bar{\Sigma}_{a,j} \Delta x_j \phi_j = \frac{1}{k} \bar{\nu}_{f,j} \bar{\Sigma}_{f,j} \Delta x_j \phi_j \ , \quad j = 1, \ldots, N \ , \tag{56}$$

where

$$\bar{\Sigma}_{a,j} = \frac{\sum_{g=1}^{G} \Sigma_{a,g,j} \phi_{g,j}}{\sum_{g=1}^{G} \phi_{g,j}} \ . \tag{57}$$

The resulting discrete one-group first-moment NDA equation is defined by

$$J_{j+1/2} = -\frac{1}{\Delta x_{j+1/2}} \left( \bar{D}^{+}_{j+1/2} \phi_{j+1} - \bar{D}^{-}_{j+1/2} \phi_j \right) \ , \tag{58}$$

where we introduce the group-averaged factors

$$\bar{D}^{+}_{j+1/2} = \frac{\sum_{g=1}^{G} D^{+}_{g,j+1/2} \phi_{g,j+1}}{\sum_{g=1}^{G} \phi_{g,j+1}} \tag{59}$$

and

$$\bar{D}^{-}_{j+1/2} = \frac{\sum_{g=1}^{G} D^{-}_{g,j+1/2} \phi_{g,j}}{\sum_{g=1}^{G} \phi_{g,j}} \ . \tag{60}$$

Equations (54) give rise to the following boundary conditions for the LONDA equations (56) and (58):

$$J_{1/2} = 0 \ , \quad J_{N+1/2} = F_X \phi_{N+1} \ , \tag{61}$$

where

$$F_X = \frac{\sum_{g=1}^{G} F_{g,X} \phi_{g,N+1}}{\sum_{g=1}^{G} \phi_{g,N+1}} \ . \tag{62}$$

Thus, the effective one-group LONDA problem is defined by Eqs. (56), (58), and (61). The MLNDA method is based on Eqs. (43), (44), (48), (49), (50), (56), (58), and (61). The iterative scheme for solving the system of equations of the MLNDA method is shown in Algorithm 3:

*Algorithm 3:* Iteration scheme of the MLNDA method

```
s = 0
• Initialization
Set D̃_g^(1/2) = 0 and F_{g,X}^(1/2) = 0.5
Define φ_g^(0) and k^(0)
• Level 1: Transport iterations
1 if ‖φ^(s) − φ^(s−1)‖_∞ > ((1/ρ_{φ,1}) − 1)ε_1
   or |k^(s) − k^(s−1)| > ((1/ρ_{k,1}) − 1)ε_1 then
      s = s + 1
      l = 0
      • Level 2: Multigroup low-order iterations
2     if ‖φ^(s,l) − φ^(s,l−1)‖_∞ > ((1/ρ_{φ,2}) − 1)ε_2
       or |k^(s,l) − k^(s,l−1)| > ((1/ρ_{k,2}) − 1)ε_2 then
         l = l + 1
         for g = 1 to G do
            Calculate B_{up,g}^(s,l−1)
         end
         for g = 1 to G do
            Solve the multigroup LONDA equations (48),
              (49), and (50) to calculate φ_g^(s,l)
         end
         for g = 1 to G do
            Calculate Σ̄_a^(s,l), Σ̄_f^(s,l), ν̄_f^(s,l), D̄^±(s−1/2,l),
              F_X^(s−1/2,l)
         end
         n = 0
         • Level 3: Newton's iterations
3        if ‖φ^(s,l,n) − φ^(s,l,n−1)‖_∞ > ε_3
          or |k^(s,l,n) − k^(s,l,n−1)| > ε_3 then
            n = n + 1
            Solve the one-group LONDA equations (56),
              (58), and (61) to calculate k^(s,l,n) and φ^(s,l,n)
         end
         φ^(s,l) = φ^(s,l,n) and k^(s,l) = k^(s,l,n)
      end
      φ_g^(s) = φ_g^(s,l), φ^(s) = φ^(s,l) and k^(s) = k^(s,l)
      Σ̄_f^(s) = Σ̄_f^(s,l), ν̄_f^(s) = ν̄_f^(s,l)
      Calculate Σ̄_{s,g}^(s)
      • Transport sweep
      for g = 1 to G do
         Solve the transport equation (43) to calculate
           ψ_g^(s+1/2)
      end
      for g = 1 to G do
         Calculate D̃_g^(s+1/2) and F_{g,X}^(s+1/2)
      end
   end
```

We also consider a modification of the MLNDA method in which the number of iterations at level 2 is not allowed



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to exceed a given value $i$. We refer to this method as MLNDA($i$). The iteration scheme for the MLNDA($i$) method is presented in Algorithm 4:

*Algorithm 4:* Iteration scheme of the MLNDA($i$) method

$s = 0$
...
• Level 1: Transport iterations
1 **if** $\|\phi^{(s)} - \phi^{(s-1)}\|_\infty > ((1/\rho_{\phi,1}) - 1)\varepsilon_1$ *or* $|k^{(s)} - k^{(s-1)}| > ((1/\rho_{k,1}) - 1)\varepsilon_1$ **then**
  $s = s + 1$
  $l = 0$
  • Level 2: Multigroup low-order iterations
2 **if** $l \le i$ *and* $\{\|\phi^{(s,l)} - \phi^{(s,l-1)}\|_\infty > ((1/\rho_{\phi,2}) - 1)\varepsilon_2$ *or* $|k^{(s,l)} - k^{(s,l-1)}| > ((1/\rho_{k,2}) - 1)\varepsilon_2\}$ **then**
    $l = l + 1$
    ...
    $n = 0$
    • Level 3: Newton's iterations
3 **if** $\|\phi^{(s,l,n)} - \phi^{(s,l,n-1)}\|_\infty > \varepsilon_3$ *or* $|k^{(s,l,n)} - k^{(s,l,n-1)}| > \varepsilon_3$ **then**
      $n = n + 1$
      ...
      $\phi^{(s,l)} = \phi^{(s,l,n)}$ and $k^{(s,l)} = k^{(s,l,n)}$
    **end**
    $\phi_g^{(s)} = \phi_g^{(s,l)}$, $\phi^{(s)} = \phi^{(s,l)}$ and $k^{(s)} = k^{(s,l)}$
    ...
  **end**
**end**

In the proposed method, the eigenvalue estimate is calculated as a solution of the one-group LONDA equations (56), (58), and (61), and hence, it is evaluated in the subspace of the lowest dimensionality. Note that this is the exact projected solution subspace, because all closure relations defined to derive the equations of the MLNDA method are exact. The one-group NDA problem is treated as the generalized eigenvalue problem in which the vector of unknowns $\vec{u}$ is formed by the grid function of the total scalar flux $\vec{\phi}$ and eigenvalue.[9] This leads to the system of nonlinear equations defined by

$$\mathcal{F}(\vec{u}) = 0 \ , \quad \vec{u} = (\vec{\phi}, \lambda) \ , \quad \lambda = \frac{1}{k} \tag{63}$$

and

$$\mathcal{F}(\vec{u}) = \begin{pmatrix} \mathcal{L}\vec{u} - \lambda\mathcal{P}\vec{u} \\ \mathcal{B}\vec{u} \\ \mathcal{N}\vec{u} - 1 \end{pmatrix} \ , \tag{64}$$

where $\mathcal{L}$, $\mathcal{P}$, and $\mathcal{B}$ are the loss, production, and boundary condition operators of the effective one-group LONDA equations and $\mathcal{N}$ is a normalization operator. The Newton's method is applied to solve the generalized eigenvalue problem (63). Thus, the one-group LONDA equations are solved by means of the following iteration process:

$$\mathcal{J}(\vec{u}^{(n-1)})\delta\vec{u}^{(n-1)} = -\mathcal{F}(\vec{u}^{(n-1)}) \ ,$$
$$\vec{u}^{(n-1)} = (\vec{\phi}^{(n-1)}, \lambda^{(n-1)}) \tag{65}$$

and

$$\vec{u}^{(n)} = \vec{u}^{(n-1)} + \delta\vec{u}^{(n-1)} \ , \tag{66}$$

where $\mathcal{J}$ is the Jacobian of $\mathcal{F}$.

The effective one-group LONDA problem (56), (58), and (61) can also be directly coupled to the multigroup high-order transport problem (43) and (44) to accelerate the transport iterations. This leads to another two-level iteration method. We refer to this method as the grey NDA (GNDA) method. The iterative scheme of the GNDA method is shown in Algorithm 5:

*Algorithm 5:* Iteration scheme of the GNDA method

$s = 0$
• Initialization
Set $\tilde{D}_g^{(1/2)} = 0$ and $F_{g,X}^{(1/2)} = 0.5$
Define $\phi_g^{(1/2)}$, $J_g^{(1/2)}$ and $k^{(0)}$
• Level 1: Transport iterations
1 **if** $\|\phi^{(s)} - \phi^{(s-1)}\|_\infty > ((1/\rho_{\phi,1}) - 1)\varepsilon_1$ *or* $|k^{(s)} - k^{(s-1)}| > ((1/\rho_{k,1}) - 1)\varepsilon_1$ **then**
  $s = s + 1$
  **for** $g = 1$ *to* $G$ **do**
    Calculate $\bar{\Sigma}_{s,g}^{(s-1/2)}$, $\bar{\Sigma}_a^{(s-1/2)}$, $\bar{\Sigma}_f^{(s-1/2)}$, $\bar{\nu}_f^{(s-1/2)}$, $\bar{D}^{\pm(s-1/2)}$, $F_X^{(s-1/2)}$ using $\phi_g^{(s-1/2)}$ and $J_g^{(s-1/2)}$
  **end**
  $n = 0$
  • Level 2: Newton's iterations
2 **if** $\|\phi^{(s,n)} - \phi^{(s,n-1)}\|_\infty > \varepsilon_3$ *or* $|k^{(s,n)} - k^{(s,n-1)}| > \varepsilon_3$ **then**
    $n = n + 1$
    Solve the one-group LONDA equations (56), (58), and (61) to calculate $k^{(s,n)}$ and $\phi^{(s,n)}$
  **end**
  $\phi^{(s)} = \phi^{(s,n)}$ and $k^{(s)} = k^{(s,n)}$
  • Transport sweep
  **for** $g = 1$ *to* $G$ **do**
    Solve the transport equation (43) to calculate $\psi_g^{(s+1/2)}$
  **end**
  **for** $g = 1$ *to* $G$ **do**
    Calculate $\phi_g^{(s+1/2)}$, $J_g^{(s+1/2)}$, $\tilde{D}_g^{(s+1/2)}$ and $F_{g,X}^{(s+1/2)}$
  **end**
**end**

## IV. NUMERICAL RESULTS

To demonstrate the performance of the proposed multilevel iterative algorithms, we consider a test



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consisting of three regions of equal size ($0 \le x \le 3H$) (Ref. 8). This is a one-dimensional version of the seven-group C5G7 benchmark.[10] The left region is a mixed oxide assembly. The middle region is a $UO_2$ assembly. The right region is a reflector. The left boundary is reflective. The right boundary is vacuum. The assembly width $H$ is 21.42 cm. There are 17 pin cells in an assembly. The spatial and angular meshes are uniform with $\Delta x = 0.09$ cm and $\Delta\mu = 0.2$, respectively. We use the following convergence criteria[3,11] for level 1:

$$\|\phi^{(s)} - \phi^{(s-1)}\|_\infty \le \left(\frac{1}{\rho_{\phi,1}^{(s)}} - 1\right)\varepsilon_1$$

and

$$|k^{(s)} - k^{(s-1)}| \le \left(\frac{1}{\rho_{k,1}^{(s)}} - 1\right)\varepsilon_1 \ , \tag{67}$$

where

$$\rho_{\phi,1}^{(s)} = \frac{\|\phi^{(s)} - \phi^{(s-1)}\|_\infty}{\|\phi^{(s-1)} - \phi^{(s-2)}\|_\infty}$$

and

$$\rho_{k,1}^{(s)} = \frac{|k^{(s)} - k^{(s-1)}|}{|k^{(s-1)} - k^{(s-2)}|} \tag{68}$$

are estimates of the spectral radius of the transport iterations by means of data available at the $s$'th iteration. The criteria for level 2 are similar to those at level 1:

$$\|\phi^{(s,l)} - \phi^{(s,l-1)}\|_\infty \le \left(\frac{1}{\rho_{\phi,2}^{(s,l)}} - 1\right)\varepsilon_2$$

and

$$|k^{(s,l)} - k^{(s,l-1)}| \le \left(\frac{1}{\rho_{k,2}^{(s,l)}} - 1\right)\varepsilon_2 \ , \tag{69}$$

where $\rho_{\phi,2}^{(s,l)}$ and $\rho_{k,2}^{(s,l)}$ are estimates of the spectral radius of iterations at level 2 at a particular transport iteration:

$$\rho_{\phi,2}^{(s,l)} = \frac{\|\phi^{(s,l)} - \phi^{(s,l-1)}\|_\infty}{\|\phi^{(s,l-1)} - \phi^{(s,l-2)}\|_\infty}$$

and

$$\rho_{k,2}^{(s,l)} = \frac{|k^{(s,l)} - k^{(s,l-1)}|}{|k^{(s,l-1)} - k^{(s,l-2)}|} \ . \tag{70}$$

The convergence criterion for the Newton's iterations at level 3 is defined as

$$\|\vec{u}^{(n)} - \vec{u}^{(n-1)}\|_\infty \le \varepsilon_3 \ . \tag{71}$$

TABLE I

Total Number of Iterations

| Iterations (Index) | Level 1: Transport Iterations, $s$ | Level 2: Multigroup Low-Order Iterations, $l$ | Level 3: Newton's Iterations, $n$ |
|---|---|---|---|
| TLNDA | 10 | 373 | — |
| MLNDA | 10 | 85 | 116 |
| GNDA | 89 | — | 171 |

The parameters for convergence criteria at various levels of iterations are $\varepsilon_1 = 10^{-8}$, $\varepsilon_2 = 10^{-9}$, and $\varepsilon_3 = 10^{-5}$. The eigenfunction is normalized such that the average value of the total scalar flux in the given spatial domain is equal to 1. The initial guess is defined by $\phi_g^{(0)} = \text{const}$ and $k^{(0)} = 1$.

The total numbers of iterations for the TLNDA, MLNDA, and GNDA methods at each level are listed in Table I. The GNDA method converges slowly. The reason is that it does not use the multigroup NDA problem. The numbers of transport iterations for the TLNDA and MLNDA methods are equal to each other. They converge in ten iterations. Both methods have the same convergence histories. In Figs. 1 and 2 we plot the convergence histories in terms of differences of eigenvalues and eigenfunctions on successive transport (level 1) iterations. The $\infty$-norm is used to compare estimates of eigenfunctions. The results presented in

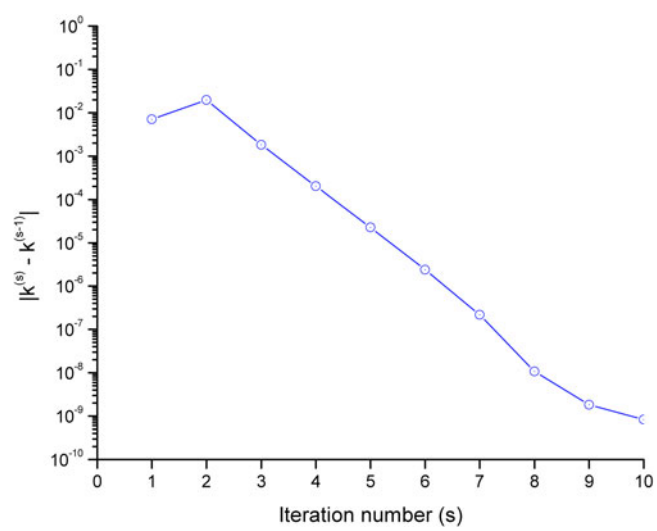


Fig. 1. Convergence histories of the eigenvalue versus the number of the transport iteration.



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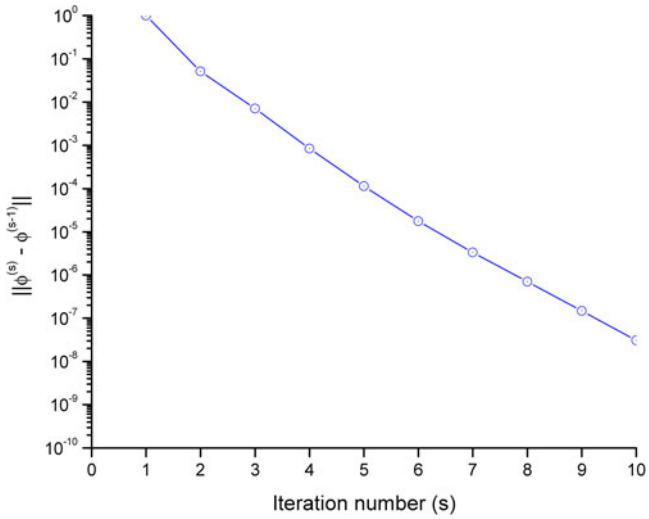


Fig. 2. Convergence histories of the eigenfunction in the $\infty$-norm versus the number of the transport iteration.

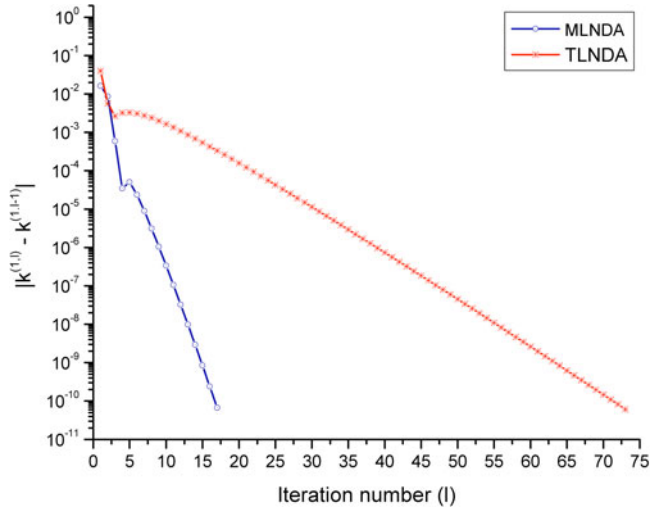


Fig. 3. $|k^{(s,l)} - k^{(s,l-1)}|$ for $s = 1$ versus the number of the multigroup low-order iteration $l$.

Table I show that the MLNDA method reduces the total number of the multigroup low-order iterations by a factor of ~4.4 in comparison with the TLNDA method. Thus, the effective one-group LONDA problem in the MLNDA method accelerates the multigroup low-order iterations. Note that the MLNDA method has certain additional computational costs associated with solving the effective one-group low-order problem. However, the amount of iterations at level 3 will not change with increase in number of groups in a problem.

Tables II and III show the number of multigroup low-order iterations on each transport iteration. The number of these iterations gradually decreases for both methods. On the first iteration ($s = 1$), the TLNDA method executed 73 multigroup low-order iterations, whereas the MLNDA method had just 17 iterations of that kind. Figures 3 and 4 present the iteration history of differences in estimations of eigenvalue and eigenfunction on the first transport iteration. Note that for $s = 1$, $\tilde{D}_g^{1/2} = 0$ (see Algorithm 3). Thus, in this case the multigroup LONDA equations are equivalent to the multigroup diffusion equation. The presented details for $s = 1$ demonstrate the efficiency of the effective one-group NDA equations in solving the eigenvalue problem of the multigroup diffusion equation. It reduces the number of multigroup low-order iterations by a factor of 4.3. Iteration histories for two various transport iterations ($s = 2$ and 4) for both methods are presented in Figs. 5 through 8,

TABLE II

Number of Inner Iterations for the TLNDA Method

| Transport iteration, $s$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Number of multigroup low-order iterations, $l$ | 73 | 62 | 56 | 47 | 40 | 32 | 25 | 19 | 12 | 7 |

TABLE III

Number of Nested Iterations for the MLNDA Method

| Transport iteration, $s$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Number of multigroup low-order iterations, $l$ | 17 | 14 | 13 | 11 | 9 | 7 | 6 | 3 | 3 | 2 |
| Number of Newton's iterations, $n$ | 29 | 22 | 18 | 14 | 11 | 8 | 6 | 3 | 3 | 2 |



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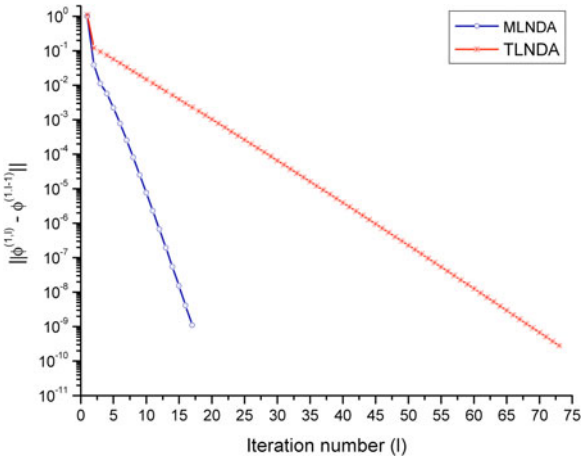


Fig. 4. $\|\phi^{(s,l)} - \phi^{(s,l-1)}\|_\infty$ for $s = 1$ versus the number of the multigroup low-order iteration $l$.

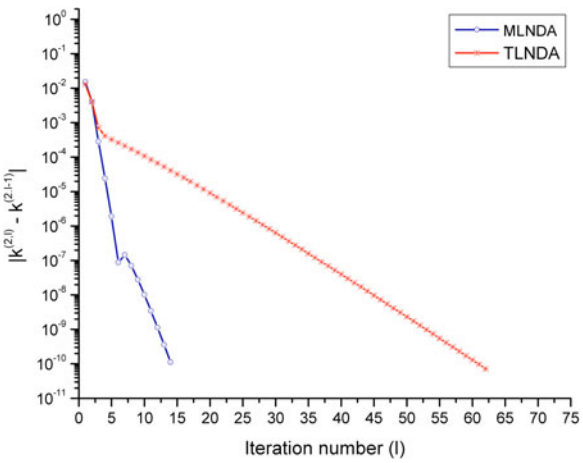


Fig. 5. $|k^{(s,l)} - k^{(s,l-1)}|$ for $s = 2$ versus the number of the multigroup low-order iteration $l$.

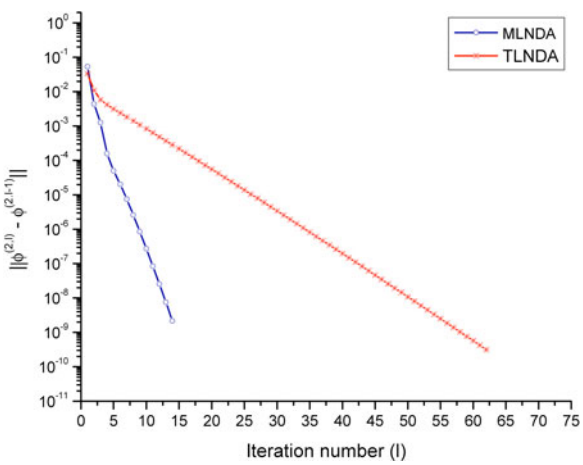


Fig. 6. $\|\phi^{(s,l)} - \phi^{(s,l-1)}\|_\infty$ for $s = 2$ versus the number of the multigroup low-order iteration $l$.

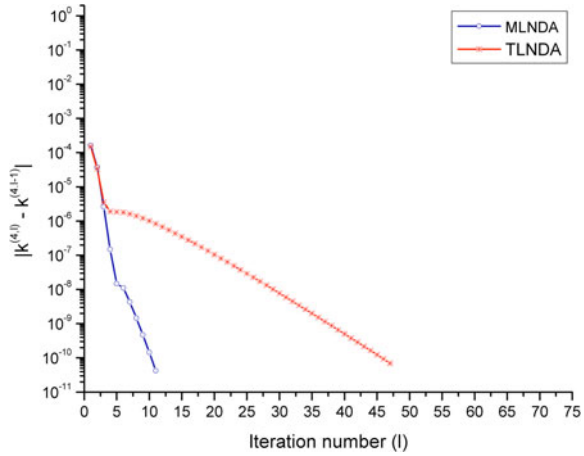


Fig. 7. $|k^{(s,l)} - k^{(s,l-1)}|$ for $s = 4$ versus the number of the multigroup low-order iteration $l$.

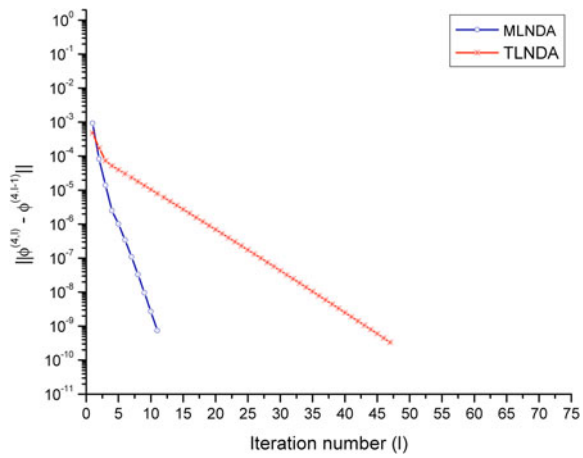


Fig. 8. $\|\phi^{(s,l)} - \phi^{(s,l-1)}\|_\infty$ for $s = 4$ versus the number of the multigroup low-order iteration $l$.

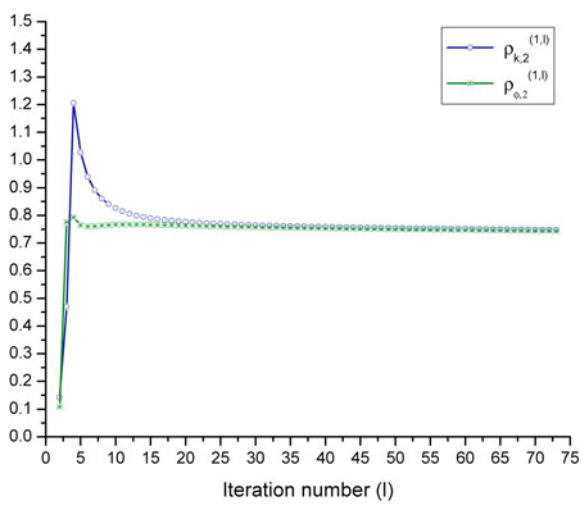


Fig. 9. $\rho_{k,2}^{(s,l)}$ and $\rho_{\phi,2}^{(s,l)}$ for the TLNDA method at $s = 1$ versus the number of the multigroup low-order iteration $l$.



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which show differences in convergence behavior of the MLNDA and TLNDA methods at level 2. Note that the TLNDA method employs a linear method of power iterations for solving the multigroup NDA equations. In the MLNDA method the iteration process at this level is a nonlinear one. Figures 9 through 12 present $\rho_{k,2}^{(s,l)}$ and $\rho_{\phi,2}^{(s,l)}$ during iterations at level 2 for $s = 1$ and 4. These data enable one to estimate approximately the spectral radii of the multigroup low-order iterations. The analysis of convergence at all transport iterations shows that in the given test the spectral radius of the multigroup low-order iterations of the TLNDA method is approximately 0.75. The spectral radius of the MLNDA method is approximately 0.27.

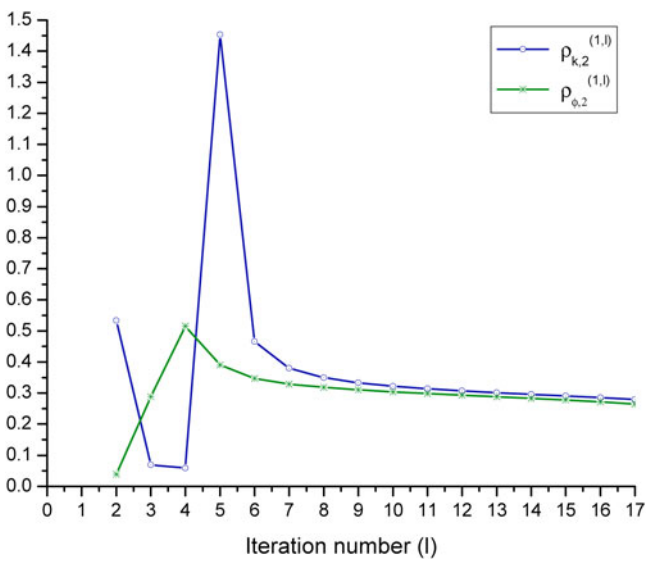


Fig. 10. $\rho_{k,2}^{(s,l)}$ and $\rho_{\phi,2}^{(s,l)}$ for the MLNDA method at $s = 1$ versus the number of the multigroup low-order iteration $l$.

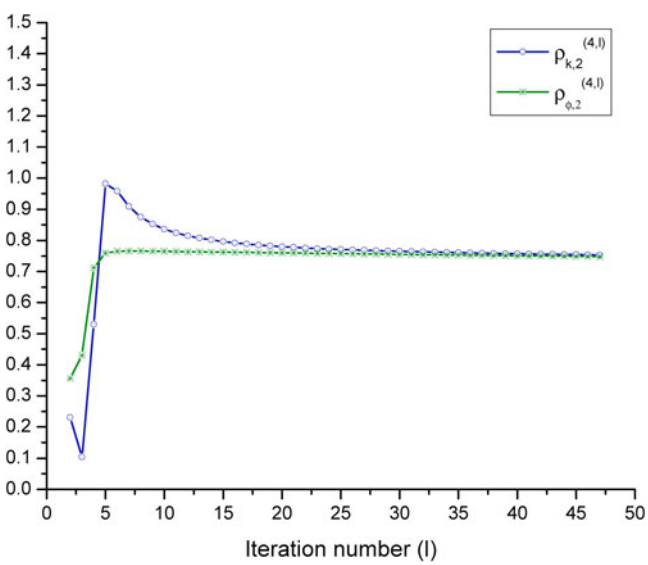


Fig. 11. $\rho_{k,2}^{(s,l)}$ and $\rho_{\phi,2}^{(s,l)}$ for the TLNDA method at $s = 4$ versus the number of the multigroup low-order iteration $l$.

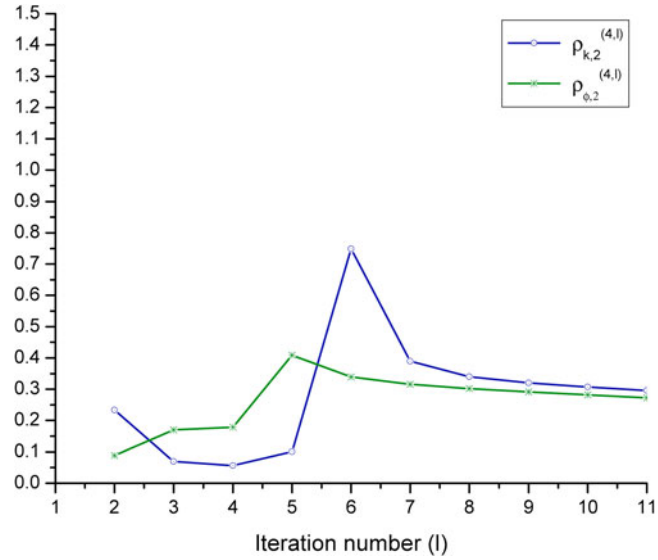


Fig. 12. $\rho_{k,2}^{(s,l)}$ and $\rho_{\phi,2}^{(s,l)}$ for the MLNDA method at $s = 4$ versus the number of the multigroup low-order iteration $l$.

We now present the results obtained with the TLNDA$(i)$ and MLNDA$(i)$ methods. On each transport iteration, the number of the multigroup low-order iterations is not allowed to be larger than a given value $i$. The number of iterations for the TLNDA$(i)$ method is listed in Table IV. The convergence histories of transport iterations are presented in Figs. 13 and 14. We notice that in the case where $i \leq 6$, the obtained numbers of transport iterations are greater than those of the TLNDA method. If $i = 7$, then the amount of transport iterations is the same as in the case where the multigroup low-order iterations are converged at each transport iteration. The total

TABLE IV

Total Number of Iterations for the TLNDA$(i)$ Method

| Iterations (Index) | Transport Iterations, $s$ | Multigroup Low-Order Iterations, $l$ |
|---|---|---|
| TLNDA(1) | 66 | 66 |
| TLNDA(2) | 33 | 66 |
| TLNDA(3) | 22 | 66 |
| TLNDA(4) | 17 | 68 |
| TLNDA(5) | 14 | 70 |
| TLNDA(6) | 11 | 66 |
| TLNDA(7) | 10 | 70 |



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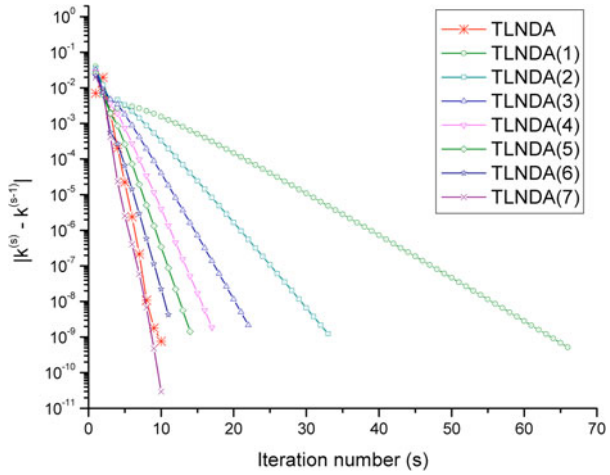


Fig. 13. Convergence histories of the eigenvalue versus the number of the transport iteration for the TLNDA$(i)$ method.

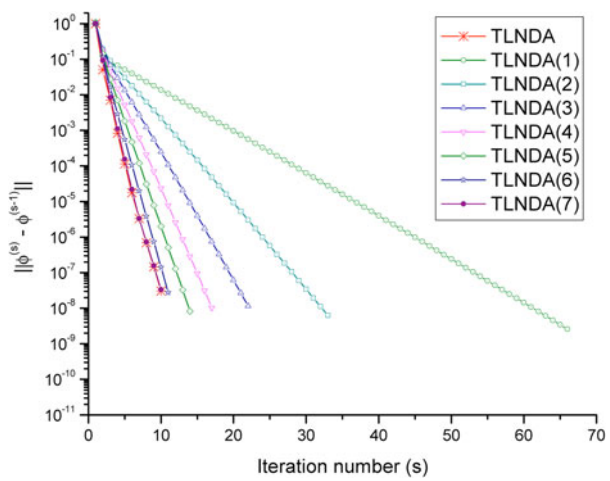


Fig. 14. $\|\phi^{(s)} - \phi^{(s-1)}\|_\infty$ versus the number of the transport iteration for the TLNDA$(i)$ method.

TABLE V

Total Number of Iterations for the MLNDA$(i)$ Method

| Iterations (Index) | Transport Iterations, $s$ | Multigroup Low-Order Iterations, $l$ | Newton's Iterations, $n$ |
|---|---|---|---|
| MLNDA(1) | 14 | 14 | 27 |
| MLNDA(2) | 10 | 20 | 37 |

number of multigroup low-order iterations of the TLNDA(7) method is less by a factor of 5.3 compared to the TLNDA method. The results for the MLNDA$(i)$ method are shown in Table V. Figures 15 and 16 demonstrate the convergence histories. This method needs to execute a maximum of two multigroup iterations to match the number of transport iterations of the MLNDA method. Moreover, the MLNDA(2) method reduces the number of multigroup low-order iterations at the same time by a factor of 3.5.

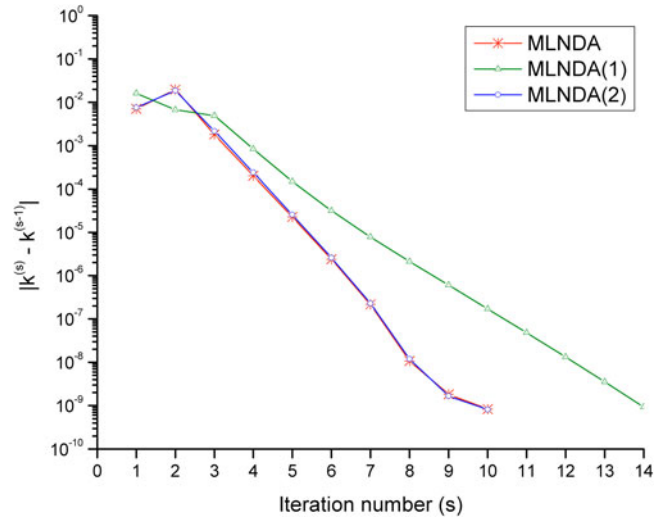


Fig. 15. Convergence histories of the eigenvalue versus the number of the transport iteration for the MLNDA$(i)$ method.

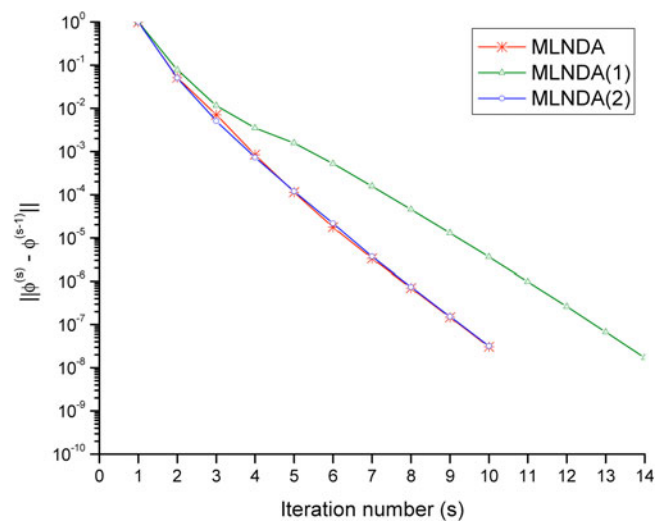


Fig. 16. $\|\phi^{(s)} - \phi^{(s-1)}\|_\infty$ versus the number of the transport iteration for the MLNDA$(i)$ method.

## V. CONCLUSIONS

New computational methods for solving the multigroup transport equation have been developed. They are multilevel iterative algorithms based on the NDA method.



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A key element of the new methods is the effective one-group LONDA problem. It is an equivalent transport eigenvalue problem in an exact solution subspace. On each transport iteration, this problem is used to estimate the eigenvalue. The one-group NDA eigenvalue problem is treated as a generalized eigenvalue problem. The Newton's method is applied to solve it. The spatial discretization of the one-group LONDA equations is consistent with the discretization of the multigroup LONDA equations. The numerical results demonstrated that the effective one-group problem accelerates multigroup low-order iterations that involve solution of the multigroup NDA equations. The proposed method can be used to solve various types of eigenvalue problems for the neutron transport equation as well as multidimensional problems. It can also be applied to the multigroup NDA equations on a spatial grid different from the one used for the multigroup transport equation to develop a multilevel coarse-mesh transport acceleration.

## ACKNOWLEDGMENTS

This research was supported by the Consortium for Advanced Simulation of Light Water Reactors, an Energy Innovation Hub for Modeling and Simulation of Nuclear Reactors under U.S. Department of Energy contract DE-AC05-00OR22725.

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