International Conference on Mathematics and Computational Methods Applied to Nuclear Science and Engineering (M&C 2011)
Rio de Janeiro, RJ, Brazil, May 8-12, 2011, on CD-ROM, Latin American Section (LAS) / American Nuclear Society (ANS)
ISBN 978-85-63688-00-2

MULTI-LEVEL NONLINEAR DIFFUSION ACCELERATION METHOD FOR MULTIGROUP TRANSPORT k-EIGENVALUE PROBLEMS

Dmitriy Y. Anistratov
Department of Nuclear Engineering
North Carolina State University
Raleigh, NC 27695-7909
anistratov@ncsu.edu

ABSTRACT

The nonlinear diffusion acceleration (NDA) method is an efficient and flexible transport iterative scheme for solving reactor-physics problems. This paper presents a fast iterative algorithm for solving multigroup neutron transport eigenvalue problems in 1D slab geometry. The proposed method is defined by a multi-level system of equations that includes multigroup and effective one-group low-order NDA equations. The eigenvalue is evaluated in the exact projected solution space of smallest dimensionality, namely, by solving the effective one-group eigenvalue transport problem. Numerical results that illustrate performance of the new algorithm are demonstrated.

Key Words: multigroup transport equation, eigenvalue problem, iterative methods.

1. INTRODUCTION

One of common iterative approaches for solving the transport equation is the nonlinear diffusion acceleration (NDA) method [1]. It is an efficient algorithm for acceleration of within-group transport iterations for reactor physics calculations [2]. The NDA method is formulated directly in the discrete form. The low-order NDA equations are defined for the moments of the transport solution and consist of (i) the particle balance equation and (ii) the first-moment equation in a form of a generalized Fick's law with a special consistency term. The high-order and low-order equations of the method are coupled by means of an exact closure. In case of energy-independent eigenvalue transport problems, the NDA method can be presented in a general differential form as follows:

$$\mu\frac{\partial\psi^{(s+1/2)}}{\partial x} + \Sigma_t\psi^{(s+1/2)} = \frac{1}{2}\left(\Sigma_s + \frac{1}{k^{(s)}}\nu_f\Sigma_f\right)\phi^{(s)} \,, \tag{1}$$

$$\phi^{(s+1/2)} = \int_{-1}^{1}\psi^{(s+1/2)}d\mu \,, \tag{2}$$

$$J^{(s+1/2)} = \int_{-1}^{1}\mu\psi^{(s+1/2)}d\mu \,, \tag{3}$$

$$\tilde{D}^{(s+1/2)} = \frac{1}{\phi^{(s+1/2)}}\left(J^{(s+1/2)} + \frac{1}{3\Sigma_t}\frac{d\phi^{(s+1/2)}}{dx}\right) \,, \tag{4}$$

$$\frac{dJ^{(s+1)}}{dx} + \left(\Sigma_t - \Sigma_s - \frac{1}{k^{(s+1)}} \nu_f \Sigma_f \right) \phi^{(s+1)} = 0 \,, \tag{5}$$

$$J^{(s+1)} = -\frac{1}{3\Sigma_t} \frac{d\phi^{(s+1)}}{dx} + \tilde{D}^{(s+1/2)} \phi^{(s+1)} \,. \tag{6}$$

The spatial grids for the transport equation (1) and low-order NDA (LONDA) equations (5) and (6) can be different. For example, the low-order NDA problem can be applied to pin-cell regions whereas the transport equation itself is solved on a mesh that resolves structure inside pin cells [2, 3]. Then, the discrete balance equation (5) in each coarse spatial cell relates the particle net leakage rate across the pin-cell boundary with absorption and fission production rates in the given pin cell. The discrete first-moment equation (6) is formulated at pin-cell interfaces. It provides a relationship between the net leakage rate at these interfaces and the average scalar fluxes in neighboring pin cells. In this paper, the low-order NDA equations are considered on grids that are used for the transport equation.

The multigroup transport equation with vacuum boundary conditions is given by

$$\begin{gathered} \mu \frac{\partial}{\partial x} \psi_g(x,\mu) + \Sigma_{t,g}(x) \psi_g(x,\mu) \\ = \frac{1}{2} \sum_{p=1}^{G} \Sigma_{s,p\to g}(x) \int_{-1}^{1} \psi_p(x,\mu) d\mu + \frac{1}{2k} \chi_g(x) \sum_{p=1}^{G} \nu_{f,p}(x) \Sigma_{f,p}(x) \int_{-1}^{1} \psi_p(x,\mu) d\mu \,, \\ -1 \le \mu \le 1, \quad 0 \le x \le X, \quad g = 1, ..., G \,, \\ \psi_g(0,\mu) = 0 \ \text{ for } \mu > 0 \,, \quad \psi_g(X,\mu) = \ 0 \ \text{ for } \mu < 0 \,. \end{gathered} \tag{7}$$

Here standard notations are used. The NDA method (1)-(6) can be used directly to solve multigroup transport problems by applying it to the transport equation in each group. In this case, the multigroup low-order NDA equations are similar in form to the multigroup P_1 equations. The method of power iterations is used to solve them. The estimation of the eigenvalue is obtained as a result of solving the multigroup low-order NDA problem. In this paper, to accelerate the convergence of multigroup transport iterations, it is proposed to formulate a one-group (grey) low-order transport problem based on the low-order NDA equations [4, 5].

The remainder of this paper is organized as follows. In Sec. 2 we present the two-level NDA method for solving multigroup transport problems. In Sec. 3 the multi-level NDA method is formulated. The numerical results are presented in Sec. 4. We conclude with a summary in Sec 5.

2. TWO-LEVEL NDA METHOD

The NDA method for solving the multigroup transport problems is defined by a high-order multigroup problem for the transport discretization scheme and a multigroup problem for the low-order NDA (LONDA) equations. To discretize the transport equation, we use the step characteristic method. The *multigroup high-order transport problem* is given by:

$$\begin{aligned} &\mu_m \big(\psi_{m,g,j+1/2} - \psi_{m,g,j-1/2} \big) + \Sigma_{t,g,j} \Delta x_j \psi_{m,g,j} \\ &\quad = \frac{\Delta x_j}{2} \left(\sum_{g'=1}^{G} \Sigma_{s,g'\to g,j} \phi_{g',j} + \frac{\chi_{g,j}}{k} \sum_{g'=1}^{G} \nu_{f,g',j} \Sigma_{f,g',j} \phi_{g',j} \right) , \end{aligned} \tag{8}$$

$$\psi_{m,g,j} = \alpha_{m,g,j}\psi_{m,g,j-1/2} + (1-\alpha_{m,g,j})\psi_{m,g,j+1/2}\,, \tag{9}$$

$$\alpha_{m,g,j} = \frac{1}{\tau_{m,g,j}} - \frac{1}{e^{\tau_{m,g,j}}-1}\,, \quad \tau_{m,g,j} = \frac{\Sigma_{t,g,j}\Delta x_j}{\mu_m}\,, \tag{10}$$

$$j = 1,...,N\,, \quad m = 1,...,M\,, \quad g = 1,...,G\,,$$

$$\psi_{m,g,1/2} = 0\,,\ m:\ \mu_m > 0\,, \tag{11}$$

$$\psi_{m,g,N+1/2} = 0\,,\ m:\ \mu_m < 0\,, \tag{12}$$

where $\psi_{g,m,j-1/2}$ and $\psi_{g,m,j}$ are the cell-edge and cell-average group angular fluxes, correspondingly,

$$\Delta x_j = x_{j+1/2} - x_{j-1/2}\,, \quad j = 1,...,N\,, \quad \Delta x_0 = \Delta x_{N+1} = 0\,. \tag{13}$$

Now the NDA method is applied to the discretized multigroup transport equation (8)-(10). Let us introduce the group scalar flux and current calculated from the group angular flux by means of quadrature sums

$$\tilde{\phi}_{g,j} = \sum_{m=1}^{M}\psi_{m,g,j}w_m\,, \quad j = 1,...,N\,, \tag{14}$$

$$\tilde{\phi}_{g,0} = \sum_{m=1}^{M}\psi_{m,g,1/2}w_m\,, \quad \tilde{\phi}_{g,N+1} = \sum_{m=1}^{M}\psi_{m,g,N+1/2}w_m\,, \tag{15}$$

$$\tilde{J}_{g,j-1/2} = \sum_{m=1}^{M}\mu_m\psi_{m,g,j+1/2}w_m\,, \quad j = 1,...,N+1\,. \tag{16}$$

These quantities are utilized to define the NDA consistency term

$$\tilde{D}_{g,j+1/2} = \frac{\tilde{J}_{g,j+1/2} + D_{g,j+1/2}\frac{(\tilde{\phi}_{g,j+1}-\tilde{\phi}_{g,j})}{\Delta x_{j+1/2}}}{0.5(\tilde{\phi}_{g,j+1}+\tilde{\phi}_{g,j})}\,, \tag{17}$$

where

$$\Delta x_{j+1/2} = \frac{1}{2}(\Delta x_j + \Delta x_{j+1})\,, \tag{18}$$

$$D_{g,j+1/2} = \frac{1}{3\Sigma_{t,g,j+1/2}}\,, \tag{19}$$

$$\Sigma_{t,g,j+1/2} = \frac{\Sigma_{t,g,j}\Delta x_j + \Sigma_{t,g,j+1}\Delta x_{j+1}}{\Delta x_j + \Delta x_{j+1}} \tag{20}$$

and define boundary factors

$$F_{g,L} = \frac{\tilde{J}_{g,1/2}}{\tilde{\phi}_{g,0}}\,, \quad F_{g,R} = \frac{\tilde{J}_{g,N+1/2}}{\tilde{\phi}_{g,N+1}} \tag{21}$$

that are used to formulate low-order boundary conditions. The resulting *multigroup LONDA equations* have the following form:

$$J_{g,j+1/2} - J_{g,j-1/2} + \Sigma_{t,g,j}\Delta x_j\phi_{g,j}$$
$$= \Delta x_j\sum_{g'=1}^{G}\Sigma_{s,g'\to g,j}\phi_{g',j} + \Delta x_j\frac{\chi_{g,j}}{k}\sum_{g'=1}^{G}\nu_{f,g'}\Sigma_{f,g'}\phi_{g',j}\,, \tag{22}$$
$$j = 1,...,N\,,$$

$$J_{g,j+1/2} = -D_{g,j+1/2}\frac{\phi_{g,j+1} - \phi_{g,j}}{\Delta x_{j+1/2}} + \frac{1}{2}\tilde{D}_{g,j+1/2}(\phi_{g,j+1} + \phi_{g,j}) \, , \quad j = 0, ..., N \, , \tag{23}$$

$$J_{g,1/2} = F_{g,L}\phi_{g,0} \, , \quad J_{g,N+1/2} = F_{g,R}\phi_{g,N+1} \, . \tag{24}$$

On each transport iteration, the equations (22)-(24) are solved by the power iteration method with the current estimates for the consistency term (17) and boundary factors (21) to evaluate the multiplication factor and associated multigroup eigenfunction. We will refer to the method described above as *two-level NDA* (TLNDA) method. The iteration scheme of the TLNDA method is presented in Algorithm 1 in a form of a pseudo code.

Algorithm 1: Iteration Scheme of the TLNDA Method

$s = 0$
for $g = 1, ..., G$ **do**
  Set $\tilde{D}_g^{(1/2)} = 0$, $F_{g,L}^{(1/2)} = -0.5$, and $F_{g,R}^{(1/2)} = 0.5$
while $||1 - \phi^{(s)}/\phi^{(s-1)}||_\infty > \varepsilon_{\phi,1}$ *or* $|1 - k^{(s)}/k^{(s-1)}| > \varepsilon_{k,1}$ **do**
  $s = s + 1$
  $l = 0$
  while $||1 - \phi^{(s,l)}/\phi^{(s,l-1)}||_\infty > \varepsilon_{\phi,2}$ *or* $|1 - k^{(s,l)}/k^{(s,l-1)}| > \varepsilon_{k,2}$ **do**
    $l = l + 1$
    for $g = 1, ..., G$ **do**
      Solve the multigroup LONDA equations (22)-(24) to calculate $\phi_g^{(s,l)}$
    Calculate $k^{(s,l)} = k^{(s,l-1)} \, ||\nu_f \Sigma_f \phi^{(s,l)}|| / \, ||\nu_f \Sigma_f \phi^{(s,l-1)}||$
  $\phi_g^{(s)} = \phi_g^{(s,l)}$ and $k^{(s)} = k^{(s,l)}$
  for $g = 1, ..., G$ **do**
    Solve the transport equation (8) to calculate $\psi_g^{(s+1/2)}$
  for $g = 1, ..., G$ **do**
    Calculate $\tilde{D}_g^{(s+1/2)}$, $F_{g,L}^{(s+1/2)}$, and $F_{g,R}^{(s+1/2)}$

---

3. MULTI-LEVEL NDA METHOD

We now formulate a new algorithm that is based on the idea of NDA method and employs extra low-order equations, namely, one-group (grey) problem for the total scalar flux ϕ and current J to accelerate multigroup transport iterations [4, 5]. Let us sum Eq. (22), (23) and (24) over groups to get

$$J_{j+1/2} - J_{j-1/2} + \Delta x_j \sum_{g=1}^{G} \left(\Sigma_{t,g,j} - \Sigma_{s,g,j}\right) \phi_{g,j} = \frac{\Delta x_j}{k} \sum_{g=1}^{G} \nu_{f,g,j} \Sigma_{f,g,j} \phi_{g,j} \, , \tag{25}$$

$$\begin{aligned} &\frac{1}{3}\left(\phi_{j+1} - \phi_j\right) - \frac{\Delta x_{j+1/2}}{2} \sum_{g=1}^{G} \Sigma_{t,g,j+1/2} \tilde{D}_{g,j+1/2} \phi_{g,j+1} \\ &- \frac{\Delta x_{j+1/2}}{2} \sum_{g=1}^{G} \Sigma_{t,g,j+1/2} \tilde{D}_{g,j+1/2} \phi_{g,j} + \Delta x_{j+1/2} \sum_{g=1}^{G} \Sigma_{t,g,j+1/2} J_{g,j+1/2} = 0 \, , \end{aligned} \tag{26}$$

$$J_{1/2} = \sum_{g=1}^{G} F_{g,L}\phi_{g,0}\,, \quad J_{N+1/2} = \sum_{g=1}^{G} F_{g,R}\phi_{g,N+1}\,. \tag{27}$$

To derive the one-group balance equation, we define averaged absorption and fission cross sections

$$\overline{\Sigma}_{a,j} = \frac{\sum_{g=1}^{G} \Sigma_{a,g,j}\phi_{g,j}}{\sum_{g=1}^{G} \phi_{g,j}}\,, \tag{28}$$

$$\overline{\Sigma}_{f,j} = \frac{\sum_{g=1}^{G} \Sigma_{f,g,j}\phi_{g,j}}{\sum_{g=1}^{G} \phi_{g,j}}\,, \tag{29}$$

and

$$\overline{\nu}_{f,j} = \frac{\sum_{g=1}^{G} \nu_{f,g,j}\Sigma_{f,g,j}\phi_{g,j}}{\sum_{g=1}^{G} \Sigma_{f,g,j}\phi_{g,j}}\,. \tag{30}$$

The equations (25), (28)-(30) lead to

$$J_{j+1/2} - J_{j-1/2} + \overline{\Sigma}_{a,j}\Delta x_j\phi_j = \frac{1}{k}\overline{\nu}_{f,j}\overline{\Sigma}_{f,j}\Delta x_j\phi_j\,, \quad j = 1, ..., N\,. \tag{31}$$

To cast Eq. (26) as an equation for ϕ and J, we define two factors

$$<\Sigma_t\tilde{D}>^-_{j+1/2} = \frac{\sum_{g=1}^{G} \Sigma_{t,g,j+1/2}\tilde{D}_{g,j+1/2}\phi_{g,j}}{\sum_{g=1}^{G} \phi_{g,j}}\,, \tag{32}$$

$$<\Sigma_t\tilde{D}>^+_{j+1/2} = \frac{\sum_{g=1}^{G} \Sigma_{t,g,j+1/2}\tilde{D}_{g,j+1/2}\phi_{g,j+1}}{\sum_{g=1}^{G} \phi_{g,j+1}}\,, \tag{33}$$

the one-group cross section given by

$$\overline{\Sigma}_{t,j+1/2} = \frac{\sum_{g=1}^{G} \Sigma_{t,g,j+1/2}|J_{g,j+1/2}|}{\sum_{g=1}^{G} |J_{g,j+1/2}|}\,, \tag{34}$$

and extra consistency terms defined as follows

$$\bar{\zeta}^+_{j+1/2} = \begin{cases} \frac{\gamma_{j+1/2}}{\sum_{g=1}^{G}\phi_{g,j+1}}\,, & \text{if } \gamma_{j+1/2} > 0 \\ 0\,, & \text{if } \gamma_{j+1/2} \leq 0 \end{cases}\,, \quad \bar{\zeta}^-_{j+1/2} = \begin{cases} 0\,, & \text{if } \gamma_{j+1/2} > 0 \\ \frac{|\gamma_{j+1/2}|}{\sum_{g=1}^{G}\phi_{g,j}}\,, & \text{if } \gamma_{j+1/2} \leq 0 \end{cases}\,, \tag{35}$$

where

$$\gamma_{j+1/2} = \sum_{g=1}^{G}\big(\Sigma_{t,g,j+1/2} - \overline{\Sigma}_{t,j+1/2}\big)J_{g,j+1/2}\,. \tag{36}$$

The resulting one-group first-moment NDA equation has the following form:

$$J_{j+1/2} = -D_{j+1/2}\frac{\phi_{j+1} - \phi_j}{\Delta x_{j+1/2}} + \frac{1}{\overline{\Sigma}_{t,j+1/2}}\Bigg(\Big(\frac{1}{2}<\Sigma_t\tilde{D}>^+_j - \bar{\zeta}^+_{j+1/2}\Big)\phi_{j+1} + \Big(\frac{1}{2}<\Sigma_t\tilde{D}>^-_j + \bar{\zeta}^-_{j+1/2}\Big)\phi_j\Bigg)\,, \tag{37}$$

where

$$D_{j+1/2} = \frac{1}{3\overline{\Sigma}_{t,j+1/2}} \,. \tag{38}$$

The equations (40) give rise to the boundary conditions for the LONDA equations (31) and (37) given by

$$J_{1/2} = F_L \phi_0 \,, \quad J_{N+1/2} = F_R \phi_{N+1} \,, \tag{39}$$

where

$$F_L = \frac{\sum_{g=1}^{G} F_{g,L}\phi_{g,0}}{\sum_{g=1}^{G} \phi_{g,0}} \,, \quad F_R = \frac{\sum_{g=1}^{G} F_{g,R}\phi_{g,N+1}}{\sum_{g=1}^{G} \phi_{g,N+1}} \,. \tag{40}$$

Thus the *effective one-group LONDA problem* is defined by Eqs (31), (37), and (39).

To couple the low-order NDA equations for ϕ and J with equations of higher dimensionality, namely,

- the multigroup equations for the group scalar flux and currents,
- the multigroup equations for the group angular flux,

and to close exactly the complete system of equations, the right-hand sides of these multigroup equations are recasted in terms of the total scalar flux. As a result, the *multigroup low-order NDA* equations for ϕ_g and J_g are defined as

$$\begin{gathered} J_{g,j+1/2} - J_{g,j-1/2} + (\Sigma_{t,g,j} - \Sigma_{s,g\to g,j})\Delta x_j \phi_{g,j} = \\ \Delta x_j \sum_{g'=1}^{g-1} \Sigma_{s,g'\to g,j}\phi_{g',j} + \Delta x_j \left(Q_{s,g,j} + \frac{\chi_{g,j}}{k}\overline{\nu}_{f,j}\overline{\Sigma}_{f,j} \right)\phi_j \,, \end{gathered} \tag{41}$$

$$j = 1, ..., N \,,$$

$$J_{g,j+1/2} = -D_{g,j+1/2}\frac{\phi_{g,j+1} - \phi_{g,j}}{\Delta x_{j+1/2}} + \frac{1}{2}\tilde{D}_{g,j+1/2}(\phi_{g,j+1} + \phi_{g,j}) \,, \tag{42}$$

$$j = 0, ..., N \,,$$

$$J_{g,1/2} = F_{g,L}\phi_{g,0} \,, \quad J_{g,N+1/2} = F_{g,R}\phi_{g,N+1} \,, \tag{43}$$

where

$$Q_{s,g,j} = \frac{\sum_{g'=g+1}^{G} \Sigma_{s,g'\to g,j}\phi_{g',j}}{\sum_{g'=1}^{G} \phi_{g',j}} \,. \tag{44}$$

The *high-order multigroup transport* equation for $\psi_{m,g}$ has the following form:

$$\mu_m\big(\psi_{m,g,j+1/2} - \psi_{m,g,j-1/2}\big) + \Sigma_{t,j}\Delta x_j \psi_{m,g,j} = \frac{\Delta x_j}{2}\left(\overline{\Sigma}_{s,g,j} + \frac{\chi_{g,j}}{k}\overline{\nu}_{f,j}\overline{\Sigma}_{f,j} \right)\phi_j \,, \tag{45}$$

$$\psi_{m,g,j} = \alpha_{m,g,j}\psi_{m,g,j-1/2} + (1 - \alpha_{m,g,j})\psi_{m,g,j+1/2} \,, \tag{46}$$

where

$$\overline{\Sigma}_{s,g,j} = \frac{\sum_{g'=1}^{G} \Sigma_{s,g'\to g,j}\phi_{g',j}}{\sum_{g'=1}^{G} \phi_{g',j}} \,. \tag{47}$$

CASL-U-2014-093-000

Finally, the *multi-level NDA (MLNDA)* method is defined by Eqs. (31), (37), (39), (41)-(43), (45), and (46). The iterative scheme for solving the system of equations of the MLNDA method is shown in Algorithm 2.

In the proposed method, the eigenvalue estimation is calculated as a solution of the one-group LONDA equations and hence it is evaluated in the subspace of the lowest dimensionality. Note that this is the exact projected solution subspace, because all closure relations defined to derive the equations of the MLNDA method are exact. The one-group NDA problem is treated as the generalized eigenvalue problem in which the vector of unknowns $\vec{u}$ is formed by the grid function of the total scalar flux ($\vec{\phi}$) and eigenvalue [6]. This leads to a system of nonlinear equations of the following form:

$$\mathcal{F}(\vec{u}) = 0\,, \quad \vec{u} = (\vec{\phi}, \lambda)\,, \quad \lambda = \frac{1}{k}\,, \tag{48}$$

$$\mathcal{F}(\vec{u}) = \begin{pmatrix} \mathcal{L}\vec{u} - \lambda \mathcal{P}\vec{u} \\ \mathcal{B}\vec{u} \\ \mathcal{N}\vec{u} - 1 \end{pmatrix} \tag{49}$$

where $\mathcal{L}$, $\mathcal{P}$, and $\mathcal{B}$ are the loss, production, and boundary condition operators of the one-group LONDA equations. $\mathcal{N}$ is the normalization operator. The Newton's method is applied to solve the generalized eigenvalue problem (48). Thus, the one-group LONDA equations are solved by means of the following iteration process:

$$\mathcal{J}(\vec{u}^{(n-1)})\delta\vec{u}^{(n-1)} = -\mathcal{F}(\vec{u}^{(n-1)})\,, \quad \vec{u}^{(n-1)} = (\vec{\phi}^{(n-1)}, \lambda^{(n-1)})\,, \tag{50}$$

$$\vec{u}^{(n)} = \vec{u}^{(n-1)} + \delta\vec{u}^{(n-1)}\,, \tag{51}$$

where $\mathcal{J}$ is the Jacobian of $\mathcal{F}$.

4. NUMERICAL RESULTS

Let us consider a test consisting of three regions of equal size ($0 \le x \le 3H$) (see Fig. 1) [5]. This is a 1D variant of the seven group C5G7 benchmark [7]. The left region is a MOX assembly. The middle one is a UO_2 assembly. The right region is a reflector. The left boundary is reflective. The right boundary is vacuum. The assembly width (H) is 21.42 cm. There are 17 pin cells in an assembly. The spatial and angular meshes are uniform with Δx =0.09 cm and $\Delta\mu$ = 0.2, correspondingly. The relative pointwise convergence criteria are used. The parameters for various levels of iterations are listed in Table I.

In this test problem, the numbers of transport iterations are the same for both methods. In Figure 2 we plot the convergence histories for the MLNDA method in terms of differences of eigenvalues and eigenfunctions on successive transport iterations. The L_2-norm is used to compare estimations in eigenfunctions. The total numbers of iterations for every level of iterations are listed in Table II. The results show that the MLNDA method reduces the number of multigroup low-order iterations, because it uses the effective one-group LONDA problem. The MLNDA method needs about 3 times less multigroup low-order iterations than the TLNDA method. Note that the MLNDA method has certain additional computational costs associated with solving the one-group low-order problem. However, the amount of one-group low-order iterations will not change with increase in number of groups.

Algorithm 2: Iteration Scheme for the Multi-Level NDA Method

$s = 0$

for $g = 1, ..., G$ **do**

  Set $\tilde{D}_g^{(1/2)} = 0$, $F_{g,L}^{(1/2)} = -0.5$, and $F_{g,R}^{(1/2)} = 0.5$

1 **while** $||1 - \phi^{(s)}/\phi^{(s-1)}||_\infty > \varepsilon_{\phi,1}$ *or* $|1 - k^{(s)}/k^{(s-1)}| > \varepsilon_{k,1}$ **do**

  $s = s + 1$

  $l = 0$

2   **while** $||1 - \phi^{(s,l)}/\phi^{(s,l-1)}||_\infty > \varepsilon_{\phi,2}$ *or* $|1 - k^{(s,l)}/k^{(s,l-1)}| > \varepsilon_{k,2}$ **do**

    $l = l + 1$

    for $g = 1, ..., G$ **do**

      Calculate $Q_{s,g}^{(s,l-1)}$

    for $g = 1, ..., G$ **do**

      Solve the multigroup LONDA equations (41)-(43) to calculate $\phi_g^{(s,l)}$

    for $g = 1, ..., G$ **do**

      Calculate $\overline{\Sigma}_a^{(s,l)}$, $\overline{\Sigma}_f^{(s,l)}$, $\overline{\nu}_f^{(s,l)}$, $\overline{\Sigma}_t^{(s,l)}$, $<\Sigma_t \tilde{D}>^{\pm(s-1/2,l)}$, $\bar{\zeta}^{\pm(s,l)}$

    $n = 0$

3   **while** $||1 - \phi^{(s,l,n)}/\phi^{(s,l,n-1)}||_\infty > \varepsilon_{\phi,3}$ *or* $|1 - k^{(s,l,n)}/k^{(s,l,n-1)}| > \varepsilon_{k,3}$ **do**

      $n = n + 1$

      Solve the one-group LONDA equations (31), (37), and (39). to calculate $k^{(s,l,n)}$ and $\phi^{(s,l,n)}$

    $\phi^{(s,l)} = \phi^{(s,l,n)}$ and $k^{(s,l)} = k^{(s,l,n)}$

  $\phi_g^{(s)} = \phi_g^{(s,l)}$, $\phi^{(s)} = \phi^{(s,l)}$ and $k^{(s)} = k^{(s,l)}$

  $\overline{\Sigma}_f^{(s)} = \overline{\Sigma}_f^{(s,l)}$, $\overline{\nu}_f^{(s)} = \overline{\nu}_f^{(s,l)}$

  Calculate $\overline{\Sigma}_{s,g}^{(s)}$

  for $g = 1, ..., G$ **do**

    Solve the transport equation (Eqs. (45) and (46)) to calculate $\psi_g^{(s+1/2)}$

  for $g = 1, ..., G$ **do**

    Calculate $\tilde{D}_g^{(s+1/2)}$, $F_{g,L}^{(s+1/2)}$, and $F_{g,R}^{(s+1/2)}$

Table I: Parameters of Convergence Criteria

Level	Type of Iterations	Iteration Index		Criterion for Eigenvalue	Criterion for Eigenfunction
		MLNDA	TLNDA		
1	Transport Iterations	s	s	$\varepsilon_{k,1}$=10^{-7}	$\varepsilon_{\phi,1}$=10^{-6}
2	Multigroup Low-Order Iterations	l	l	$\varepsilon_{k,2}$=10^{-8}	$\varepsilon_{\phi,2}$=10^{-7}
3	One-group Low-Order Iterations	n	–	$\varepsilon_{k,3}$=10^{-9}	$\varepsilon_{\phi,3}$=10^{-8}

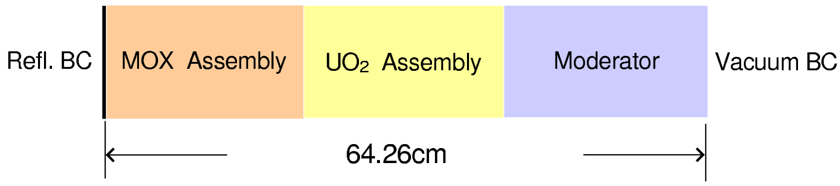

Figure 1: Test problem.

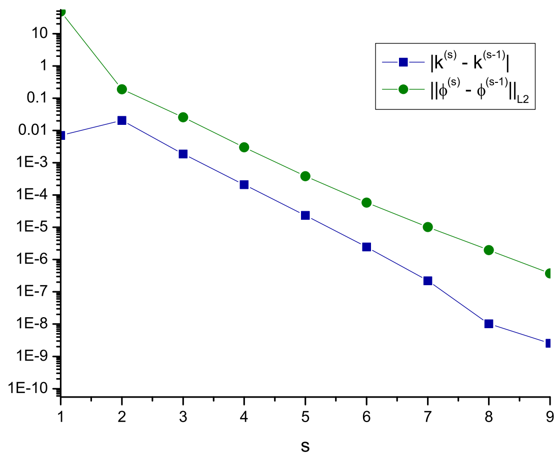

Figure 2: Convergence histories for the MLNDA method versus the number of the transport iterations.

Tables III and IV present the number of multigroup low-order iterations on each transport iteration. The numbers of these iterations gradually decrease for both methods. On the first transport iteration (s=1), the TLNDA methods executed 52 multigroup low-order iterations, whereas the MLNDA had just 16 iterations of that kind. Figures 3 and 4 show the iteration history of differences in estimations of eigenvalue and eigenfunction on the first transport iteration. Note that for s=1 $\tilde{D}_g^{1/2}$=0 (See Algorithm 2). Thus, in this case the multigroup low-order NDA equations are equivalent to the multigroup diffusion equation. The presented details for s=1 demonstrate the efficiency of the one-group NDA equations in solving the eigenvalue problem of the multigroup diffusion equation. It reduces the number of multigroup low-order iterations by a factor of 3.25.

Table II: Total Number of Iterations for TLNDA and MLNDA

Iterations (*Index*)	Transport Iterations (s)	Multigroup Low-Order Iterations (l)	One-Group Low-Order Iterations (n)
TLNDA	9	221	–
MLNDA	9	76	169

Table III: Number of Inner Iterations for TLNDA

Transport Iteration (s)	1	2	3	4	5	6	7	8	9
Number of Multigroup Low-Order Iterations (l)	56	45	38	30	22	13	4	3	1

Table IV: Number of Nested Iterations for MLNDA

Transport Iteration (s)	1	2	3	4	5	6	7	8	9
Number of Multigroup Low-Order Iterations (l)	16	15	12	10	8	6	4	3	2
Number of One-Group Low-Order Iterations (n)	39	35	27	22	16	12	8	6	4

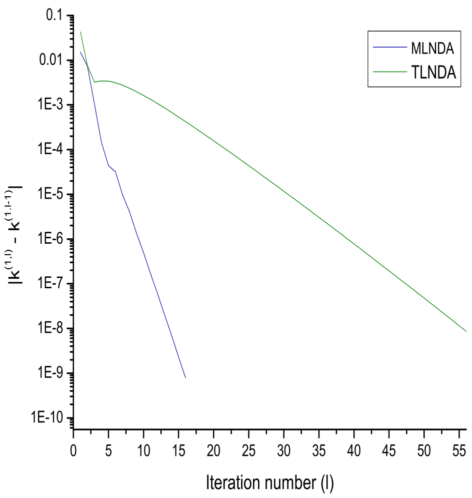

Figure 3: $|k^{(s,l)} - k^{(s,l-1)}|$ for $s = 1$ versus the number of multigroup low-order iterations (l).

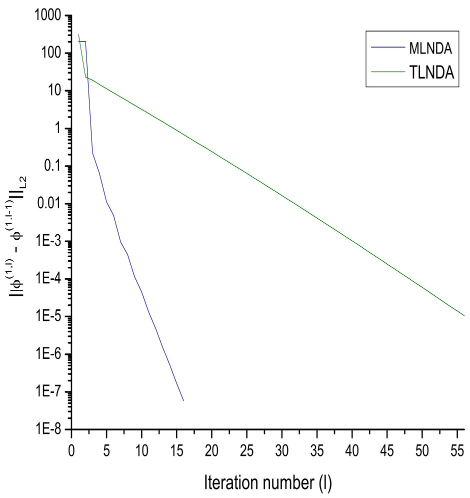

Figure 4: $||\phi^{(s,l)} - \phi^{(s,l-1)}||_{L_2}$ for $s = 1$ versus the number of multigroup low-order iterations (l).

5. CONCLUSIONS

A new computational method for solving the multigroup transport equation has been developed. This is a multi-level iterative algorithm that is based on the nonlinear diffusion acceleration method. A key element of the new method is the effective one-group low-order NDA problem that is an equivalent transport eigenvalue problem in exact solution subspace for the total scalar flux and current. On each transport iteration, this problem is used to estimate the eigenvalue. The one-group NDA eigenvalue problem is treated as a generalized eigenvalue problem. The Newton's method is applied to solve it. The spatial discretization of the one-group LONDA equations is consistent with the discretization of the multigroup LONDA equations. The numerical results demonstrated that the effective one-group problem accelerates multigroup low-order iterations that involve solution of the multigroup NDA equations. The proposed method can be used to solve various types of eigenvalue problems for the neutron transport equation. It can also be extended to multidimensional problems.

ACKNOWLEDGEMENTS

This research was supported by the Consortium for Advanced Simulation of Light Water Reactors (www.casl.gov), an Energy Innovation Hub (http://www.energy.gov/hubs) for Modeling and Simulation of Nuclear Reactors under U.S. Department of Energy Contract No. DE-AC05-00OR22725.

REFERENCES

1. K. S. Smith, "Nodal Method Storage Reduction by Nonlinear Iteration," *Trans. Am. Nucl. Soc.*, **44**, 265 (1984).

2. K. S. Smith and J. D. Rhodes, III, "Full-Core, 2-D, LWR Core Calculations with CASMO-4E," *Proceedings of PHYSOR 2002*, Seoul, Korea, October 7-10, 2002.

3. D.A. Knoll, H. Park, and K.S. Smith, "Application of the Jacobian-Free Newton-Krylov Method to Nonlinear Acceleration of Transport Source Iteration in Slab Geometry," *Nuclear Science and Engineering* (2011) (to appear).

4. D. Y. Anistratov and V. Ya. Gol'din, "Solution of the Multigroup Transport Equation by the Quasi-Diffusion Method," *Preprint of the Keldysh Institute for Applied Mathematics*, the USSR Academy of Sciences, No. 128, 19 pp. (1986) (in Russian).

5. D.Y. Anistratov & V. Ya. Gol'din, "Multi-Level Methods for Solving Multigroup Transport Eigenvalue Problems in 1D Slab Geometry," *Proceedings of M&C 2009, International Conference on Advances in Mathematics, Computational Methods, and Reactor Physics,* Saratoga Springs, NY, May, 2009, 12 pp.

6. D. F. Gill and Y. Y. Azmy, "Newton's Method for Solving k-Eigenvalue Problems in Neutron Diffusion Theory," *Nucl. Sci. Eng.,* **167**, pp. 1-13 (2011) (to appear),

7. E. E. Lewis, M. A. Smith, N. Tsoulfanidis, G. Palmiotti, T. A. Taiwo & R. N. Blomquist. "Benchmark Specification for Deterministic 2-D/3-D MOX Fuel Assembly Transport Calculations without Spatial Homogenization (C5G7)," Expert Group on 3-D Radiation Transport Benchmarks, NEA/NSC/DOC(2001)4